Institute of Actuaries of Australia

Risk Measures and Capital for Dependent Risks

Prepared by Florence Wu and Michael Sherris

Presented to the Institute of Actuaries of Australia
Financial Services Forum 2006
11 and 12 May, 2006

The Institute of Actuaries of Australia
Level 7 Challis House 4 Martin Place
Sydney NSW Australia 2000
Telephone: +61 2 9233 3466 Facsimile: +61 2 9233 3446
Email: actuaries@actuaries.asn.au Website: www.actuaries.asn.au

Risk Measures and Capital for Dependent Risks

Florence Wu [a,*], Michael Sherris [b,1]

[a] *Ph D Candidate, Actuarial Studies, Faculty of Commerce and Economics, MetLife Insurance Limited, Level 9, 2 Park Street, Sydney, NSW, 2000, Australia*

[b] *Actuarial Studies, Faculty of Commerce and Economics, University of New South Wales, Sydney, NSW, 2052, Australia*

This version: 27 April, 2006

Abstract

Economic capital has become an important component of bank and insurance companies' financial and risk management. Risk measures and risk based capital have been the focus of increased research arising from regulatory requirements such as Basel II and the insurer risk based capital requirements. Copulas are increasingly used for modelling the dependence structure of financial and insurance risks. Traditional risk measures for insurance and financial risks based on multivariate normal distributions may not measure the dependence of many insurance risks, partly because of the tail dependence in these risks. This paper assesses the use of copulas as a modelling approach for dependent risks. We consider commonly used risk measures for economic capital. Our aim is to quantify the impact on risk measures and capital requirements of different copula based bivariate dependent risk models. We use commonly proposed copula functions and estimate VaR and TailVaR (CTE) risk measures used for economic capital assuming different levels of dependence and different marginal distributions. We consider the diversification benefits arising from different levels of dependence. We determine and report the standard errors of the economic capital estimates. Our experiments with bivariate copula models provide guidance in the practical application of dependent risk models. Surprisingly, for the sample size we assume, we find that the level of dependence and the marginal distributions have more of an effect on risk measures than the copula used, except for the TailVaR risk measure at high probability levels. Taking into account the standard errors of our estimates, it is difficult to distinguish between the estimated risk measures for different copulas.

Keywords: Economic Capital, Value at Risk (VaR), Tail Value at Risk (TailVaR), Conditional Tail Expectation (CTE), Dependence, Copula
JEL classification: C13, C16

* Corresponding author.
E-mail address: email: florence_wu3@yahoo.com (Florence Wu) m.sherris@unsw.edu.au (Michael Sherris)

1 Sherris acknowledges funding support from Australian Research Council Discovery Grants DP0345036 and DP0663090 and from the UNSW Actuarial Foundation of the Institute of Actuaries of Australia.

1 Introduction

In recent years, economic capital has become a major focus of bank and insurer risk management. Determining an adequate level of risk based capital for regulatory purposes is also very important for banks and insurers. Much of the use of economic capital has been driven by changes in banking and insurer prudential regulations. These regulations require the banks and insurance companies to assess the risks of their lines of business and other activities, and to hold sufficient capital against these risks. Internal models are allowed under Basel II and generally there is an increased emphasis on risk based assessment of capital. Increasingly, economic capital is being used for financial decision making such as pricing, assessing business performances and assessment of profitability of lines of business, products and divisions.

The commonly used approach to assessing economic capital has been the Value at Risk approach, usually denoted as VaR risk measure. This is a frequency of loss risk measure based on a quantile of the distribution of the risk under consideration. The risk could be an exposure to a market traded financial instrument or an insurance product. Other risk measures such as Tail Value at Risk, often referred to as TailVaR and also known as Conditional Tail Expectation (CTE), also allows for the severity of the potential loss beyond the Value at Risk. TailVaR is increasingly used for economic capital. There are many other risk measures that have been proposed. However, the most commonly used measures continue to be VaR and TailVaR.

Multivariate risk models incorporating dependence, and in particular copulas, are important in risk modelling in the banking and insurance areas. Copulas are one of the favored models for dependence in risk management. Recent literature demonstrates that linear correlation assumptions that have been the standard assumptions for many financial modelling including portfolio theory, option pricing and asset allocation, are misleading when risks are not normally distributed. Furthermore, this assumption is also misleading when dependence structures are nonlinear. Embrechts et al (1999)[8] demonstrate how linear correlation can be a source of confusion. They show that it is possible to have two different multivariate distributions with the same marginal distributions and the same linear correlation but with quite different dependencies.

The shortcomings of linear correlation to measure nonlinear risk dependencies is a critical issue to an insurer. This is because insurance risks have been shown to exhibit positive tail dependence that is not linear, see for example Venter (2002) [16]. This tail dependence has a significant impact on the assessment of the capital requirements for an insurance company as demonstrated in, amongst others, Tang (2005) [14] and Tang and Valdez (2005) [15].

Copulas are being increasingly used in the market, credit and operational risk models in banking and insurance. In insurance risk modelling and capital management, the increased use of stochastic modelling for capital assessment that is mainly based on Dynamic Financial Analysis (DFA), has increased interest and research on the use of copulas in modelling dependencies by line of business. Practitioners and researchers have been applying copulas in their DFA models to capture the dependence structure of an insurance portfolio. Isaacs (2003) [11] uses a Gumbel copula to model the multivariate loss distribution and to consider the capital requirements of a general insurer and compares the capital requirements based on a Gumbel copula with that determined using model based on linear correlations. Tang and Valdez (2005) [15] investigate the impact of the use of t-copulas with varying degrees of freedom on the economic capital of a general insurer and evaluate the impact of dependence on the diversification benefits in a multi-line insurer.

Multivariate modelling of risks in insurance is an area that has not been explored in much detail and the impact of different dependence structures on economic capital is not well understood. The quantification of the effect of dependence structure on risk measures and capital is an important first step in the application of more advanced models in practical applications. We assess the effect of dependence using experiments with different univariate marginal distributions and different copulas at different levels of dependence between risks. We focus on bivariate risks since this highlights the impact the selection of different copulas most clearly. We use commonly adopted copulas in both finance and insurance dependence modelling.

2 Risk Measures and Economic Capital

We will investigate the impact of different dependence between risks on the economic capital of an insurer using copulas in the bivariate case by considering the two commonly used risk measures; Value-at-Risk (VaR) and Tail Value-at-Risk (TailVaR). The VaR risk measure has been adopted by the Basel Committee on Banking Supervision for assessing the capital adequacy requirements for banks and is also a commonly used risk measure for setting risk limits in banks. In insurance, the probability of ruin has been a concept that actuaries and regulators have studied and this is equivalent to the VaR risk measure. We begin with a brief coverage of the basic concepts of the two risk measures.

2.0.1 Value at Risk

Value-at-Risk (VaR) is one of the main risk measurement methods used by practitioners to calculate the capital requirements of a risk portfolio. There are weaknesses in this risk measure. Artzner (1999) [1] provides a discussion of these weaknesses. Despite these weaknesses identified by researchers, VaR continues to be the most commonly used method for assessing risk capital and has become a standard measure used in determining economic capital. The definition of VaR is the maximum loss borne by a financial institution over a specified period of time at a specified probability, α, that is, Pr(Loss $>$ VaR) $=$ 1 - α. If X is an (absolutely) continuous random variable whose cumulative distribution function $F(X)$ describes the negative value of a risky financial position at a specified horizon time t, then for a probability $0 < \alpha < 1$, $\text{VaR}_{\alpha}(\text{X}) = \inf\{\text{x}|\text{F(X)} > \alpha\}$. In general, α is between 0.95 and 0.995 depending on the time horizon and the application.

2.0.2 Tail Value at Risk

Since VaR only takes into account the probability of a loss it takes no account of the size of the loss in the event that the loss exceeds the VaR. It is also not a coherent risk measure because it is not sub-additive which means that the sum of the VaR measures on a portfolio of risks can exceed the VaR risk measure determined for the total portfolio. Portfolio risk measures are expected to be less than or equal to the sum of the risk measures for the individual risks in a portfolio because of the effect of diversification on the risk measure. Tail Value at Risk, also called TailVaR, is an extension of VaR. TailVaR allows for the severity of the loss. It has the advantage over VaR that it satisfies the properties of a coherent risk measure and is sub-additive. It is a more conservative measure than VaR since it takes into account the average of the losses that exceed VaR, resulting in a higher risk measure.

If X is an (absolutely) continuous random variable with distribution function $F(x)$ that describes the negative profit and loss distribution of the risky financial position at the specified horizon time τ (thus losses are positive), then the TailVaR for X is (Bradley (2003) [2])

$$\mathrm{TailVaR}_\alpha(\mathrm{X}) = \mathrm{E}(\mathrm{X}|\mathrm{X} > \mathrm{VaR}_\alpha(\mathrm{X})).$$

There are various risk measures that are the same as or equivalent to TailVaR. For example Conditional Tail Expectation (CTE) is equivalent to TailVaR.

3 Copulas

Copulas are a useful technique for modelling multivariate risks. They allow the multivariate distribution to be separated into the marginal distributions and a function that captures the multivariate dependence between the marginal distributions. Copulas are multivariate distribution functions whose one-dimensional marginals are uniform on the interval [0,1]. We provide a brief review of some basic concepts of copulas and how they measure dependence. Further details are found in many sources including Nelsen (1999) [13] and Cherubini et al (2004) [3].

Assume a portfolio of d risks each with continuous strictly increasing distribution functions. The joint probability distribution for a portfolio of risks is denoted by

$$F_X(x_1, \ldots, x_d) = \Pr(X_1 \leq x_1, \ldots, X_d \leq x_d)$$

In practice data is often analyzed for the marginal distributions of each risk, or a parametric form for these distributions is assumed and fitted. The marginal distributions are denoted by $F_{X_1}, \ldots, F_{X_d}$ where $F_{X_i}(x) = \Pr(X_i \leq x_i)$.

The joint probability distribution can be written as

$$
\begin{aligned}
F_{\mathbf{X}}(x_1,\ldots,x_d) &= \Pr(X_1 \le x_1,\ldots,X_d \le x_d) \\
&= \Pr(F_1(X_1) \le F_1(x_1),\ldots,F_d(X_d) \le F_d(x_d)) \\
&= \Pr(U_1 \le F_1(x_1),\ldots,U_d \le F_d(x_d))
\end{aligned}
$$

where each U_i is uniform $(0,1)$. From this we derive the underlying form of a copula function. Sklar's Theorem (Nelsen 1999 [13]) is an important result that shows the existence of the copula function and the relationship between the univariate margins and the multivariate distribution function. We will consider only the case of continuous marginal distributions. This Theorem shows that any continuous multivariate distribution has a unique copula given by

$$
F_{\mathbf{X}}(x_1,\ldots,x_d) = C(F_1(x_1),\ldots,F_d(x_d))
$$

For discrete distributions the copula exists but may not be unique. The converse result is that the copula can be written in terms of the multivariate distribution and the univariate margins as

$$
C(u_1,\ldots,u_d) = F_{\mathbf{X}}\left(F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)\right)
$$

where $F_i^{-1}(u_i)$ are the inverses, or quantiles, of the margins, assumed continuous and non-decreasing.

Another important result is that if $(X_1,\ldots,X_d)$ have copula C and if $T_1,\ldots,T_d$ are increasing continuous functions then $(T_1(X_1),\ldots,T_d(X_d))$ also has copula C. The relevance of this result is that, if a copula model applies to specified risks, then a log (or similar) transformation of the risks will have the same copula or dependence structure.

Properties of the copula include $P[V \le v|U \le u] = \frac{\partial C(u,v)}{\partial u}$ and $P[V > v|U = u] = 1 - \frac{C(u,v)}{\partial u}$, and $P[U > u|V = v] = 1 - \frac{C(u,v)}{\partial v}$.

Archimedean copulas are a special class of copulas that are widely used in finance and insurance because they are simple to construct and there are many convenient properties possessed by this class of copulas. Nelsen (1999) ([13]) provides details of these and a good discussion is also given in Frees and Valdez (1998) [9]. This class of copulas is defined by a generator φ, a convex decreasing function with domain $(0,1]$ and range $[0,\infty)$ such that $\varphi(1) = 0$, with

$$
C_\varphi(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))
$$

with u,v in (0,1]. A number of the copulas we will examine come from this popular class of copulas.

We will consider only the bivariate case. In general we will have many more risks to consider and the computational and estimation issues will be increased as a result. Consider the bivariate cumulative distribution $F(x,y) = C(F_1(x),F_2(y))$ then the density is given by

$$
f(x,y) = \frac{\partial F(x,y)}{\partial x \partial y}
$$

$$= f_1(x) f_2(y) \frac{\partial C(F_1(x), F_2(y))}{\partial x \partial y}$$
$$= f_1(x) f_2(y) c(F_1(x), F_2(y))$$

where c is the density of C.

Linear correlation is not sufficient as a measure of dependence for most risks in finance and insurance. Other measures of dependence including for example rank correlation are more useful measures of non-linear relationships. For X and Y with joint distribution F and copula C and marginal distribution functions F_X and F_Y, (X_1, Y_1) and (X_2, Y_2) independent and identically distributed pairs of random variables each with joint distribution F, then Kendall's Tau is defined as

$$\tau = \Pr[(X_1 - X_2)(Y_1 - Y_2) > 0] - \Pr[(X_1 - X_2)(Y_1 - Y_2) < 0]$$

and Spearman's Rho, which measures the linear correlation of $F_X(X), F_Y(Y)$, is defined as

$$\rho_S(X, Y) = \rho(F_X(X), F_Y(Y))$$

where $\rho(u, v)$ is the linear correlation. Spearman's Rho is also equal to

$$\rho_S(X, Y) = 12 \int_0^1 \int_0^1 [C(u, v) - uv] \, du dv$$

so that it can be interpreted as a measure of the distance between the actual distribution and the case of independence since $C(u, v) = uv$ is the independence copula.

There are many copulas that have been proposed for use in insurance and financial risk models. Nelsen (1999) [13] provides an excellent coverage of methods of generating copulas as well as many of the copula functions.

4 Dependence for Copula Models

Insurance risks exhibit tail dependence that is not linear (Venter (2002) [16]). We briefly review the tail dependence of some commonly used copula functions. Details of the tail dependence measure for copulas are provided in the literature in papers such as Embrechts et al (2001) [7], Nelsen (1999) [13], and Lindskog (2000) [12].

A commonly used definition of tail dependence is as follows.

Definition 1 *Let X and Y be continuous random variables with distribution functions F_1 and F_2. The coefficient of upper tail dependence of X and Y is*

$$lim_{u \to 1} P[Y > F_2^{-1}(u) | X > F_1^{-1}(u)] = \lambda_U$$

where $\lambda_U \in$ *[0,1] exists. If* $\lambda_U \in (0,1]$, *X and Y are said to be asymptotic dependent in the upper tail; if* $\lambda_U = 0$, *X and Y are said to be asymptotic independent in the upper tail.*

The lower tail dependence can be defined in a similar way as

$$lim_{u \to 0+} P[Y <= F_2^{-1}(u) | X <= F_1^{-1}(u)] = \lambda_L$$

This is asymptotic tail dependence and can be used to characterize different bivariate copulas. In fact we have for bivariate copulas:

Definition 2 Upper tail dependence of copulas. *If C is a bivariate copula such that*

$$lim_{u \to 1^-} \bar{C}(u,u)/(1-u) = \lambda_U$$

exists, then C has upper tail dependence if $\lambda_U \in (0,1]$, *and no upper tail dependence if* $\lambda_U = 0$. *Note that* $\bar{C}(u,u) = 1 - 2u + C(u,u)$.

Definition 3 Lower tail dependence of copulas. *If C is a bivariate copula such that*

$$lim_{u \to 0^+} \bar{C}(u,u)/(u) = \lambda_L$$

exists, then C has lower tail dependence if $\lambda_L \in (0,1]$, *and no lower tail dependence if* $\lambda_L = 0$.

5 Tail dependence of bivariate copulas

This section compares the tail dependence of four of the commonly used copulas that we will adopt for our study. Details of the properties of these copulas are given in the literature including Nelsen (1999) [13], Cherubini (2004) [3] and Embrechts et al (2001)[7]. The distribution functions and densities of the copulas we consider areshown in Figure 2.

Gumbel-Hougaard Copula

The Gumbel-Hougaard, or Gumbel, copula is one of most commonly use copula functions for measuring insurance risks because of its extreme value nature. As described in Cherubini et al (2004) ([3], page 127), "the lower and upper bound for its parameter correspond to the product copula and the upper Fréchet bound." It has the flexibility to handle independence and positive dependence.

The form of the copula is

$$C_\theta(u,v) = exp(-[(-lnu)^\theta + (-lnv)^\theta]^{\frac{1}{\theta}}),$$

for $\theta > 1$

$$\frac{\bar{C}(u,u)}{1-u} = \frac{1-2u+C(u,u)}{1-u}$$

$$= \frac{1-2u+exp(2^{1/\theta} lnu)}{1-u}$$

$$= \frac{1-2u+u^{2^{1/\theta}}}{1-u}$$

By l'Hopital rule, the upper tail dependence is given by

$$\lambda_U = lim_{u\to 1^-} \bar{C}(u,u)/(1-u) = 2 - lim_{u\to 1^-} 2^{1/\theta} u^{2^{1/\theta - 1}}$$

$$= 2 - 2^{1/\theta}.$$

Thus for $\theta > 1$, $\lambda \in (0,1]$, $C(u,u)$ has upper tail dependence. To show that $\lambda_U \in (0,1]$ for $\theta > 1$, we can first check the lower limit by setting $\lambda_U = 0$. We can see that $\lambda = 1$ when $\theta = 1$. We can also show that, if $\theta \to \infty$, we will have $\lambda_U = 1$. Hence, $\lambda_U \in (0,1]$ with $\theta > 1$, and the Gumbel copula has upper tail dependence. The Gumbel copula does not exhibit lower tail dependence.

Frank Copula

The Frank copula is a commonly used copula by researchers including Venter (2002) [16], Frees and Valdez (1999) [9]. It is a member of the class of Archimedean copulas. This is the only Archimedean copula that has the property of being radially symmetric. Radial symmetry is defined next and discussed in more detail in Nelsen (1999) [13].

Definition 4 Radial Symmetry *Let X and Y be continuous random variables with joint distribution function H marginal distribution functions F and G, respectively, and copula C. Further suppose that X and Y are symmetric about a and b, respectively. Then (X,Y) is radially symmetric about (a,b) so that*

$$H \text{ satisfies } H(a+x, b+y) = \bar{H}(a-x, b-y) \text{ for all } (x,y) \in \Re^2$$

if and only if C satisfies the following.

$$C(u,v) = \hat{C}(u,v) \text{ for all } (u,v) \in \boldsymbol{I}.$$

Hence radial symmetry implies that the pairs (u,v) and (1-u,1-v) have the same Frank distribution. The Frank copula can be generated in a simple manner because it is an Archimedean copula and can be defined in terms of a generator.

Definition 5 *Suppose u and v are random variables with copula C(u,v) with parameter α > 0 generated by $\phi(t)$ = -ln((e$^{-\theta t}$-1)/(e$^{-\theta}$-1)). Then the Frank copula C is defined as follows.*

$$C(u,v) = -\frac{1}{\theta} ln1 + \frac{(e^{u\theta}-1)(e^{u\theta}-1)}{e^{-\theta-1}}$$

The inverse of the generator can be expressed as $\phi^{-1}(s) = 1\ /\ \theta\ ln(1+e^{\theta s}(e^{-\theta}-1))$ and is completely monotonic for $\theta > 0$. The Frank copula has the following features.

$$C_\theta(u,v) = -\frac{1}{\theta} ln(1+\frac{(e^{\theta u}-1)(e^{\theta v}-1)}{(e^{\theta}-1)}$$

$$\frac{\bar{C}(u,u)}{1-u} = \frac{1-2u+C(u,u)}{1-u}$$

$$\lambda_U = lim_{u\to 1^-}\bar{C}(u,u)/(1-u) = lim_{u\to 1^-}(1+2u+\frac{1}{\theta}ln(1+\frac{(e^{\theta u}-1)^2}{(e^{\theta}-1)}))/(1-u)$$

So that $lim_{u\to 1^-}\bar{C}(u,u)/(1-u) = 0$ and, as discussed in Venter (2002) [16], the Frank copula does not have upper tail dependence. Since it is radially symmetric we immediately have $\lambda_U = \lambda_L = 0$, so that it has no lower tail dependence. The Frank copula is not a copula that can capture extreme tail dependence for risk measures in contrast to the Gumbel copula.

Gaussian Copula

The Gaussian copula is widely used by both practitioners and researchers in both the finance and insurance industry. The dependence of this copula is fully determined by parameters for a correlation matrix. It is has become a standard model in the area of credit risk for modelling dependence between company asset returns or default times.

The bivariate copula is given by the function

$$C(u,v) = \int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} exp(\frac{s^2-2\rho st+t^2}{2(1-\rho^2)})dsdt,$$

where $-1 < \rho < 1$ and Φ is the univariate standard normal distribution function.

The upper tail dependence is given by

$\lambda_U = lim_{u\to 1^-}\bar{C}(u,u)/(1-u)$

$= -lim_{u\to 1^-}\frac{d\bar{C}(u,u)}{du} = 2lim_{u\to 1^-}P[V > u|U = u] = lim_{u\to 1^-}P[\Phi^{-1}(V) > x|\Phi^{-1}(U) > y] = 0$

so that the Gaussian copula has no upper tail dependence.

Student-t Copula

The Student-t copula is another family of copulas that is increasingly used in finance and insurance. As with the Gaussian copula, the dependence for these copulas is fully captured by the parameters of a correlation matrix but the Student-t copula has the benefit of having (upper and lower) tail dependence in contrast to the Gaussian copula that has no tail dependence. It is important to note that, although the Gaussian copula has no tail dependence, the definition of tail dependence used is an asymptotic measure. So this is expected to be most relevant for very high probabilities when determining VaR and TailVaR risk measures.

The Student-t copula is given by

$$C(u,v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi\sqrt{(1-\rho^2)}}(1+\frac{s^2-2\rho st+t^2}{\nu(1-\rho^2)})dsdt$$

The upper tail dependence is given by

$$\lambda_U = lim_{x\to\infty} P(X > x|Y > y)$$

$$= 2lim_{x\to\infty}\bar{t}_{\nu+1}((\tfrac{\nu+1}{\nu+x^2})^{1/2}\tfrac{x-\rho x}{\sqrt{1-\rho^2}})$$

$$= 2\bar{t}_{\nu+1}(\tfrac{\sqrt{(\nu+1)}\sqrt{1-\rho}}{\sqrt{1+\rho}})$$

Details of the tail dependence measures described above can be seen in Embrechts et al (2001) [7] and Lindskog (2000)[12].

Table 1 below compares the upper and lower tail dependence measures of the copulas we will use in our experiments. To summarise:

(1) Both Gumbel and Student-t copulas have non-zero upper tail dependence.
(2) Both Gaussian and Frank copulas have zero upper tail dependence.

6 Bivariate Risk Models using Copulas

We will assess these four commonly used copulas as suitable candidates for assessing risk based capital. We will use different marginal distributions along with these different copulas to assess the impact of each of these. We will also assume different levels of dependence. We are

Copula	Upper Tail Dependence	Lower Tail Dependence
Gumbel	$\lambda_U = 2 - 2^{1/\theta}$ for $\theta > 1$	$\lambda_L = 0$
Frank	$\lambda_U = 0$	$\lambda_L = \lambda_U = 0$
Gaussian	$2lim_{x\to\infty}\bar{\Phi}(x\sqrt{(1-\rho)}/(\sqrt{(1-\rho^2)}))$ i.e. $\lambda_U = 0$ for $\rho < 1$. Hence, no upper tail dependence.	$\lambda_L = \lambda_U = 0$
Student t Copula	$2\bar{t}_{\nu+1}(\frac{\sqrt{(\nu+1)}\sqrt{1-\rho}}{\sqrt{1+\rho}})$ ν = degrees of freedom, ρ = correlation	$\lambda_L = \lambda_U$

Table 1
Tail dependence properties of the copulas used in this study.

interested in assessing which of these factors is important in assessing economic capital. Table 2 summarises the characteristics of the four copulas.

Copula	Characteristics	Dependence Measure
Guassian	Symmetric higher probability mass concentrated at the center. Does not capture tail dependence. -1 $\leq \tau \leq$ 1	$\tau = 2 * \frac{asin(r)}{\pi}$ $\rho = 6 * \frac{asin(\frac{r}{2})}{\pi}$ r = linear correlation and $-1 \leq r \leq 1$
Student-t	Symmetric Related to the Gaussian copula Captures both left and right tail dependence higher probability mass concentrated at the center. -1 $\leq \tau \leq$ 1	$\tau = 2 * \frac{asin(r)}{\pi}$ $\rho = 6 * \frac{asin(\frac{r}{2})}{\pi}$ r = linear correlation and $-1 \leq r \leq 1$
Gumbel	Right tail extreme, "comet-like" shape Not suitable for random variables that are negatively correlated $\theta \in [1, \infty]$ Boundaries: $C_1 = \Pi$, C_∞ = M $0 \leq \tau \leq 1$	$\tau_\theta = 1 - \frac{1}{\theta}$ no closed form for the Spearman's rho
Frank	Radially symmetric Capture both left and right tail dependence $\theta \in$ (-∞, ∞)\ 0 Boundaries and special case: $C_{-\infty}$ = W, $C_0 = \Pi$, C_∞ = M -1 $\leq \tau \leq$ 1	τ_θ = 1 - $\frac{4}{\theta}$(1 - $D_1(\theta)$) ρ_θ = 1 - $\frac{12}{\theta}$($D_1(\theta)$ - $D_2(\theta)$) $D_k(x) = \frac{k}{x}\int \frac{x^k}{e^t-1}$dt $D_k(-x) = D_k(x) + \frac{kx}{k+1}$

Table 2
Summary of characteristics of the copulas used in the experiments in this study.

To illustrate the bivariate models we will use in this study, we simulated 1000 samples for each of the bivariate copulas using MatLab. Figure 1 shows the resulting dependence structure of the copulas at various levels of dependence. Figure 2 shows the copulas and the density of the copulas with Kendall's correlation equal to 0.5.

In Figure 1:

- row 1 shows the Gaussian copula;
- row 2 shows the Student-t copula with 2 degrees of freedom;
- row 3 shows the Gumbel copula; and

- row 4 shows the Frank copula.
- column 1 assumes ρ = 0.5 for Gaussian and Student-t copulas, θ = 1.5 for Gumbel copula and θ = 3.5 for Frank copula. This is equivalent to Kendall's tau of 0.71.
- column 2 assumes ρ = 0.9 for Gaussian and Student-t copulas, θ = 3.5 for Gumbel copula and θ = 10.2 for Frank copula. This is equivalent to Kendall's tau of 0.3333.
- column 3 assumes ρ = -0.5 for Gaussian and Student-t copulas, no negative correlation for Gumbel copula and θ = -3.5 for Frank copula. This is equivalent to Kendall's tau of -0.71.
- column 4 assumes ρ = -0.9 for Gaussian and Student-t copulas, no negative correlation for Gumbel copula and θ = -10.2 for Frank copula. This is equivalent to Kendall's tau of -0.3333.

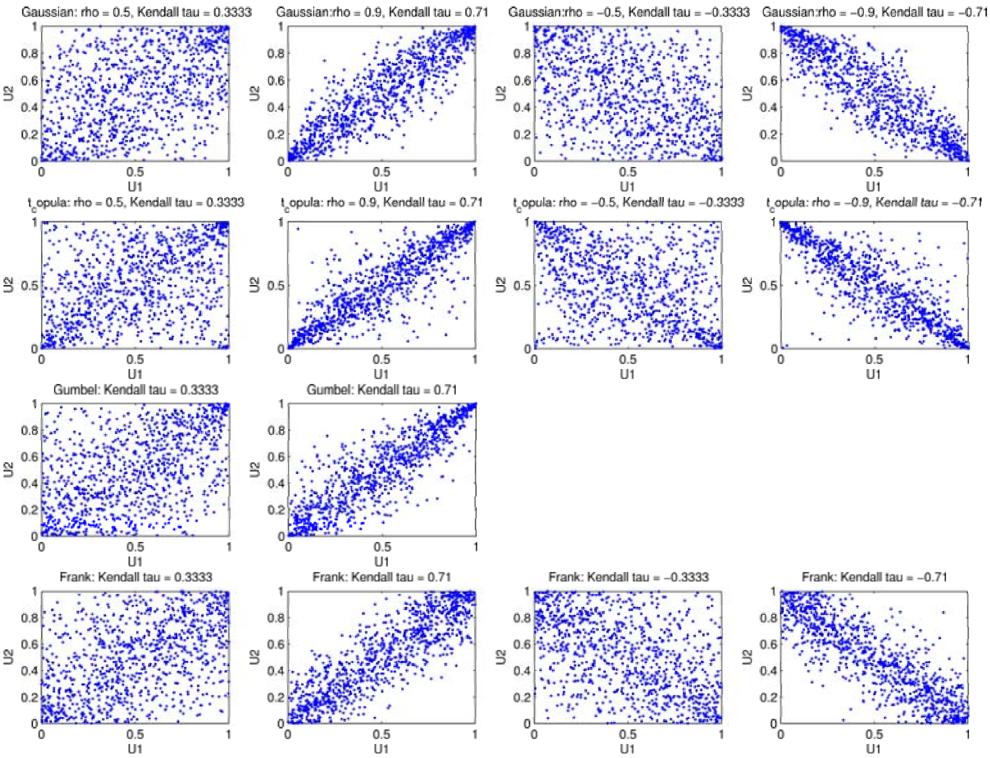


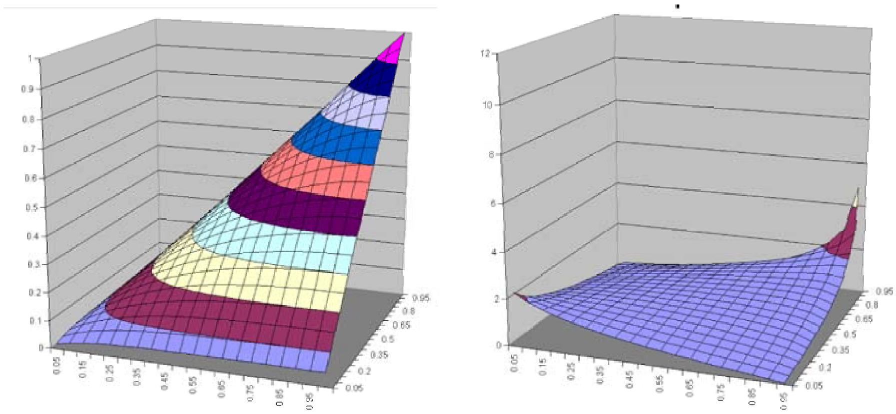
Fig. 1. *Illustration of the impact of dependence for the bivariate copulas.*

Figures 1 and 2 illustrate the following important features of these bivariate copulas.

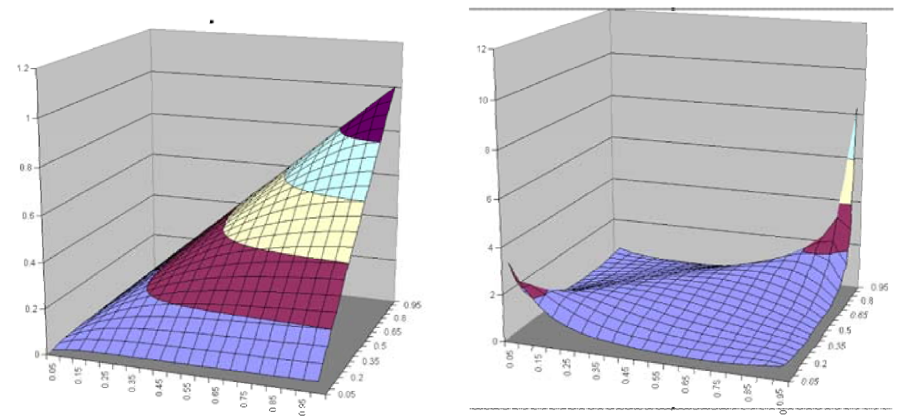
(1) Both Gaussian and Student-t copulas are symmetric. The Student-t copula has fatter tails than the Gaussian copula.
(2) The Frank copula is radially symmetric.
(3) The Gumbel copula only models positive dependence.

6.1 Marginal Distributions

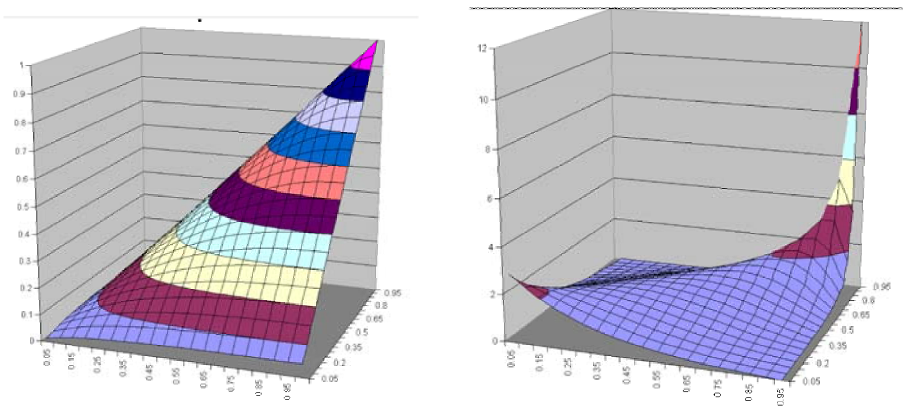
Figures 3 to 5 show the distribution of the pair-wise observations of the four copulas with different marginal distributions. We use Normal, Lognormal and Gamma marginal distributions



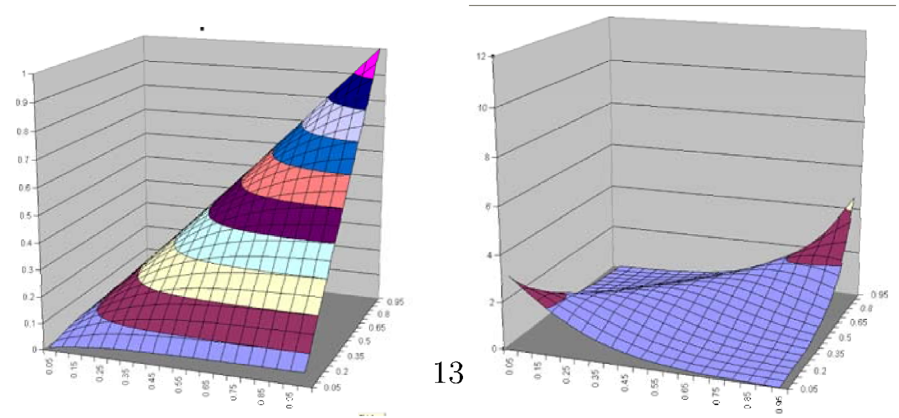
Gaussian Copula



Student-t Copula



Gumbel Copula



Frank Copula

Fig. 2. *Copulas and the Density of the Copulas*

since these are often used in financial and insurance risk modelling. It is important to consider the assessment of economic capital in terms of the full distribution of risks allowing for both the marginal distribution, the level of dependence and the dependence structure.

- column 1 shows $\rho = 0.5$ for Gaussian and Student-t copulas, $\theta = 1.5$ for Gumbel copula and $\theta = 3.5$ for Frank copula. This is equivalent to Kendall's tau of 0.71.
- column 2 shows $\rho = 0.9$ for Gaussian and Student-t copulas, $\theta = 3.5$ for Gumbel copula and $\theta = 10.2$ for Frank copula. This is equivalent to Kendall's tau of 0.3333.
- column 3 shows $\rho =$ -0.5 for Gaussian and Student-t copulas, no negative correlation for Gumbel copula and $\theta =$ -3.5 for Frank copula. This is equivalent to Kendall's tau of -0.71.
- column 4 shows $\rho =$ -0.9 for Gaussian and Student-t copulas, no negative correlation for Gumbel copula and $\theta =$ -10.2 for Frank copula. This is equivalent to Kendall's tau of -0.3333.

6.1.1 Normal Marginal Distribution

Figure 3 shows the distribution of the pair-wise observations of the four copulas at various levels of dependence assuming that the underlying marginal distributions are from a Normal distribution. As an example, this could be for a model of portfolio returns since these bivariate distributions generalise the bivariate normal distribution to include different levels of dependence including tail dependence. For the purpose of comparison, we assume that the marginal distributions have both mean and variance equal to 1. Figure 3 illustrates that,

- As levels of dependence increases, the Gaussian copula has most of the observations at the origins and toward the diagonals of the two-dimensional space. The distribution of the observations of the Gaussian copula exhibits an oval shape.
- The Student-t copula has most of the observations concentrated at the diagonals and with some observations at the tails.
- The Gumbel copula exhibits a "comet-like" shape.
- The Frank copula exhibits a more "rectangular-like" shape at higher levels of dependence.

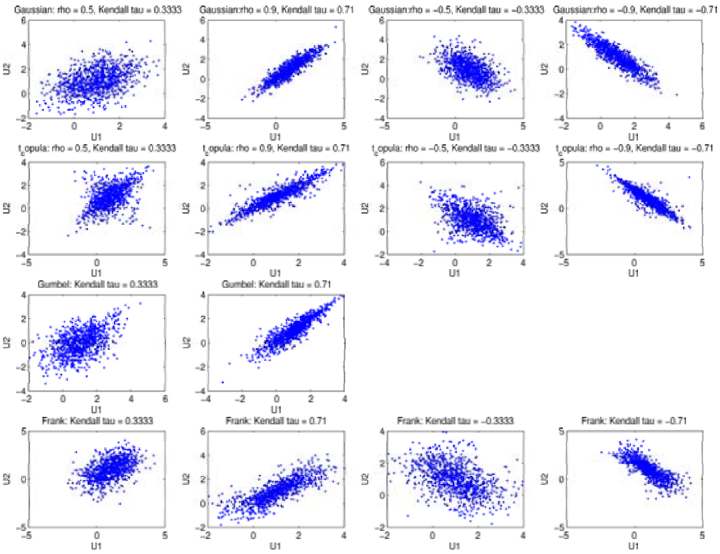


Fig. 3. *Dependence structure of the copulas with Standard Normal marginals.*

6.1.2 Lognormal Marginal Distribution

Figure 4 shows the distribution of the pair-wise random observations of the four copulas at different levels of dependence assuming Lognormal marginal distributions. For the purposes of comparison, we assume that the marginal distributions have both mean and variance equal 1. The main features to note are:

(1) Positive dependence gives "comet" shape plots.
(2) As the levels of dependence increase, the random samples are more concentrated on the positive diagonal with different shapes for different copulas. For example, the top end of the Frank copulas are relatively wider spread than the other copulas.
(3) For negative dependency, the observations produce a "boomerang" shape.

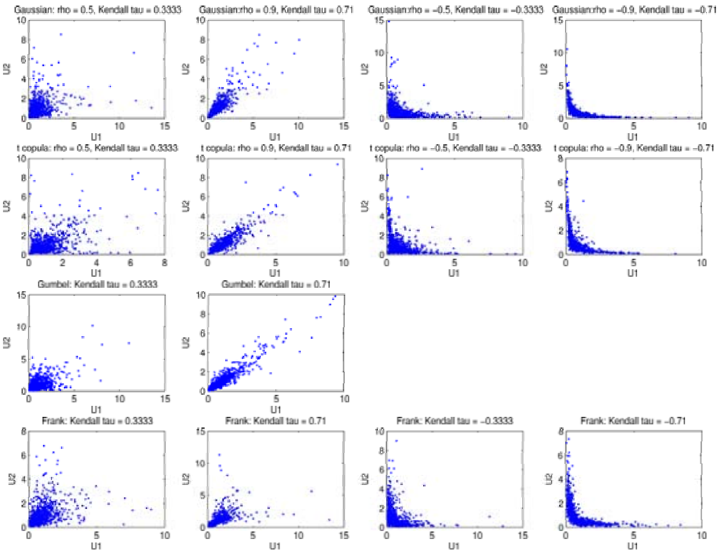


Fig. 4. *Dependence structure of the copulas with Lognormal marginals.*

6.1.3 Gamma Marginal Distribution

Figure 5 shows the distribution of the observations of the four copulas assuming that the underlying marginal distributions are Gamma. The Gamma distribution has similar properties to the Lognormal. In general, the Gamma and Lognormal distributions are similar except in the extreme tails. For the purpose of illustration, we assume that the marginal distributions have both mean and variance equal to 1. The main features to note are:

- The graphs are very similar to those with Lognormal marginals.
- The distribution of the observations are more widely spread at the origin than than for Lognormal marginals.
- "Comet" and "boomerang" shapes are similar but more prevalent than for Lognormal marginals.

It is clear from this analysis that the marginal distribution has an effect on the bivariate distribution as would be expected. The analysis of multivariate data needs to consider both the marginal distribution and the copula.

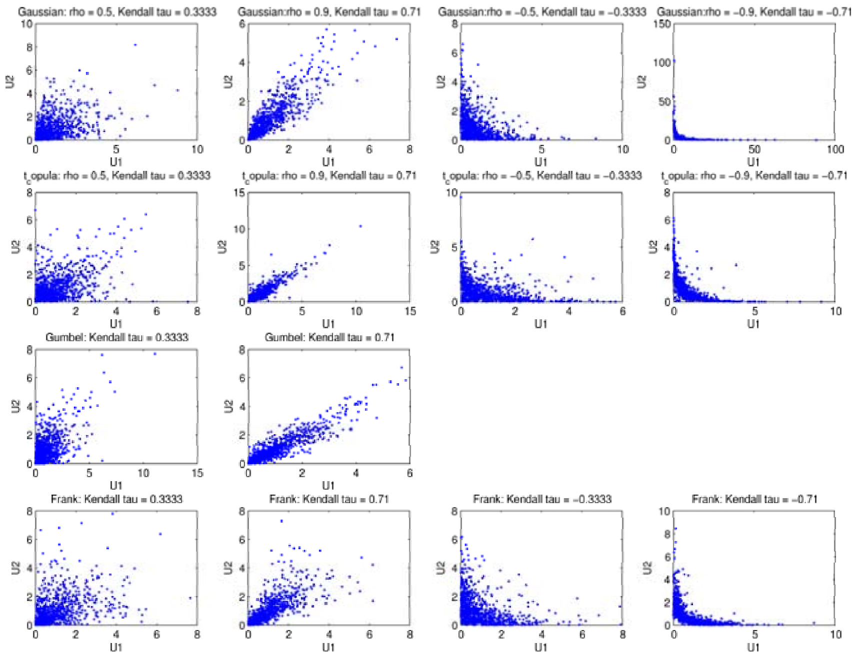


Fig. 5. *Dependence structure of the copulas with Gamma marginals.*

7 Assessing Risk Measures and Economic Capital

Understanding the extent to which dependence should be allowed for in assessing risk and in determining economic capital for measures such as VaR and TailVaR is clearly important in the risk and prudential capital management of insurers and banks. In order to assess the impact of different copulas and marginal distributions we use an experimental simulation study approach. We assess both VaR and TailVaR risk measures.

For our experiments, we consider the bivariate case and assume that the marginal distributions of the two risks are identical. We consider positive dependence since insurance risks are expected to have positive dependence. Since we are using simulation, we determine the standard errors (SE) of the estimated values of the risk measures for each experiment using a bootstrapping method. To do this we re-simulate 1000 times each of the simulations. For each simulation, we generate 1000 samples. The assumptions used in the experiments are summarised below. Table 6 sets out the assumptions for the parameters of each of the experiments. We assess the risk measures for the sum of the two risks in a portfolio.

The bivariate models assessed in this study use the following assumptions:

- Independent with Lognormal marginal distributions.
- Independent with Gamma marginal distributions.
- Positive dependence with Lognormal marginal distributions.
- Positive dependence with Gamma marginal distributions.
- Strong positive dependence with Lognormal marginal distributions.

- Strong positive dependence with Gamma marginal distributions.

Experiment	Linear Correlation	Kendall correlation	Parameters
Zero dependence	0	0	mean = 1, variance = 1, Gamma marginal: $\alpha = 1$, $\beta = 1$, Lognormal marginal: μ = -0.34657359, σ = 0.83255461, Gumbel parameter = 1, Frank parameter = 0.5, t-copula: degree of freedom = 2
Positive dependence	0.5	0.3333	mean = 1, variance = 1, Gamma marginal: $\alpha = 1$, $\beta = 1$, Lognormal marginal: μ = -0.34657359, σ = 0.83255461, Gumbel parameter = 1.5, Frank parameter = 3.3057, t-copula: degree of freedom = 2
High positive dependence	1.0	0.9991	mean = 1, variance = 1, Gamma marginal: $\alpha = 1$, $\beta = 1$, Lognormal marginal: μ = -0.34657359, σ = 0.83255461, Gumbel parameter = 12.95, Frank parameter = 40, t-copula: degree of freedom = 2

Fig. 6. Assumptions for each experiment.

7.1 Standard Errors

Since we use a simulation method, we also assess the accuracy of our simulated values by reporting estimated standard errors. Hardy (2005) [10] discusses the standard errors of the simulated VaR and TailVaR. Since both VaR and TailVaR are asymptotically unbiased, the bias is reduced by using a high number of simulations. Hardy (2003)[10] reports different methods for assessing the standard errors of estimates for economic capital risk measures.

7.1.1 Standard Error for VaR

For VaR, we can determine the variance of the estimated VaR using a normal approximation to the binomial distribution as below. The $\beta - confidence$ interval for the true $\alpha - quantile$ is (VaR_{j-k},VaR_{j+k}), where j = n $\times$ α and k is determined from the cumulative normal distribution using:

$$\beta = 2\Phi(\frac{k}{\sqrt{n\alpha(1-\alpha)}}) - 1. \tag{1}$$

α	0.9	0.92	0.95	0.99
k	19	17	13	6
j+k	919	937	963	996
j-k	881	903	937	984

Table 3
k values for confidence intervals for VaR for different α probabilities (with $\beta = 0.95$).

7.1.2 Standard Error for TailVaR

For TailVaR, Hardy (2005) [10] notes that the standard errors can be determined using the influence function approach. The influence function estimate of the standard error of the $TailVaR_\alpha$ is the standard error of the mean of the sample of $n(1-\alpha)$ worst outcomes adjusted for the uncertainty in the quantile and is given by

$$\frac{W^* + \alpha(TailVaR^*_\alpha - Q^*_\alpha)^2}{n(1-\alpha)}. \tag{2}$$

where $L_{(j)}$ = The j-th quantile of the simulated distribution.
$TailVaR^*_\alpha$ = the $\alpha - TailVaR$.
$Q^* = \alpha - quantile$ measure.
W^* is the variance of the estimator given by,

$$W^* = \frac{\sum_{j=n\alpha+1}^{n} (L_{(j)} - TailVaR^*_\alpha)^2}{n(1-\alpha) - 1}. \tag{3}$$

and

$$TailVaR^*_\alpha = \frac{\sum_{j=n\alpha+1}^{n} L_{(j)}}{n(1-\alpha)}. \tag{4}$$

7.1.3 Bootstrap

Another way to estimate the standard errors is to use a bootstrap approach. This is done by re-running a large number of simulations. For example, for our estimates, we generate the distributions of VaR and TailVaR by running 1000 simulations. We then estimate the 95% confidence intervals for the values of VaR and TailVaR using re-sampling. The approximate confidence intervals for VaR and TailVaR based on the bootstrap approach used are $(\mu_{VaR} - 2\sigma_{VaR}, \mu_{VaR} + 2\sigma_{VaR})$ and $(\mu_{TailVaR} - 2\sigma_{TailVaR}, \mu_{TailVaR} + 2\sigma_{TailVaR})$ where the standard deviations for the risk measures are approximated empirically from the re-sampled estimates of the risk measures. This is computationally intensive and a comparison with the analytical approach outlined in Hardy (2005) [10] will provide useful information about the relative accuracy of these two approaches to estimating the standard errors of the estimated risk measures.

7.2 Zero Dependence

We assess different levels of dependence. In the case of zero dependence, we compare the values of VaR and TailVaR for different copulas assuming that the random variables have zero dependence. Tables 4 to 6 give the graphical and numerical comparisons of the values of VaR and TailVaR. The results of the experiments are:

(1) With zero dependence, the results show that the Student-t copula produces the highest estimated values of VaR and TailVaR particularly for TailVaR at the higher probability levels.
(2) As expected the estimated values of VaR and TailVaR increases with the probability level α.
(3) The standard errors of the simulations increase with the probability level used for the risk measure. Estimation error (standard errors) are important in assessing risk measures and for the sample size used in these studies they are high, making it difficult to distinguish statistically between the different copulas in most cases.
(4) Allowing for the standard errors, there is very little difference between the risk measures. The copula is not as critical as the risk measure used for this level of zero dependence.

7.3 Moderate Dependence

In this section we compare the values of VaR and TailVaR for different copulas where the risks have positive dependence. Tables 7 to 9 give the graphical and numerical comparisons of the estimates of VaR and TailVaR. The results of the experiments are:

(1) As with the zero dependence case, the difference between the risk measures do not vary much between the different copulas. The copula is not as important as the level of dependence or the marginal distribution in estimating the risk measure.
(2) The standard errors are in general higher than for the case with zero dependence.
(3) The standard errors of the simulations increase with the probability level used in the risk measure. The estimation of the standard error is important.
(4) As with zero dependence, the values of VaR and TailVaR increases with the probability level α as expected.

7.4 High Dependence

Tables 10 to 12 gives the graphical and the numerical comparisons for the high dependence case. In some cases, the differences are more significant that those with lower dependence levels. The results of the experiments are:

(1) As with the cases of zero dependence and moderate dependence, the values of VaR and

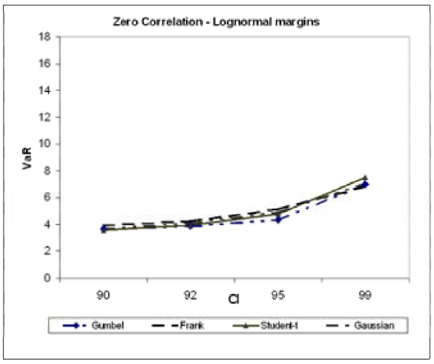
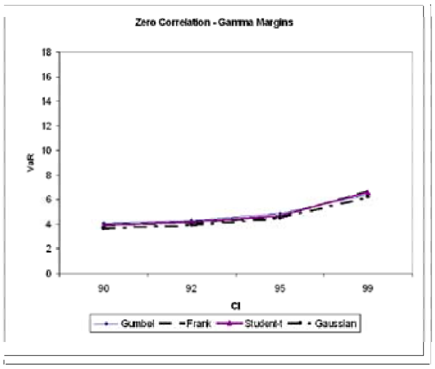
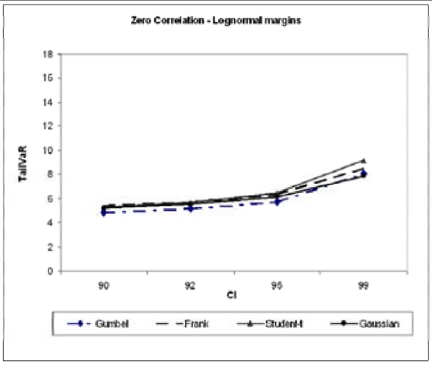
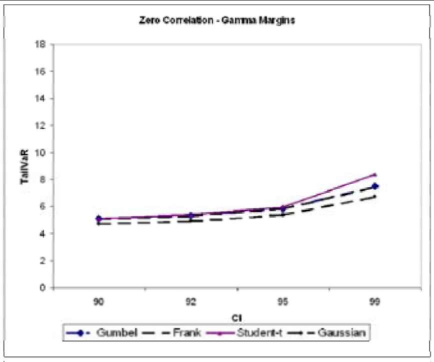
<table>
<tr><td></td><td>Lognormal</td><td>Gamma</td></tr>
<tr><td>VaR</td><td>
</td><td>
</td></tr>
<tr><td>TailVaR</td><td>
</td><td>
</td></tr>
</table>

Table 4
Zero Dependence with Lognormal and Gamma marginals.

Copula:	Gumbel	Gumbel SE	Frank	Frank SE	Student-t	Student-t SE	Gaussian	Gaussian SE
VaR(90)	3.656	0.1258	3.9531	0.1355	3.5869	0.1314	3.7242	0.1253
VaR(92)	3.9171	0.1431	4.2645	0.1551	3.9557	0.1504	4.121	0.1433
VaR(95)	4.3645	0.2004	5.1553	0.2111	4.8063	0.2198	4.9404	0.1937
VaR(99)	6.6686	0.5608	6.7314	0.5876	7.3445	0.7654	6.9539	0.5874
TailVaR(90)	4.864	0.4396	5.4041	0.4309	5.2661	0.6579	5.1992	0.4251
TailVaR(92)	5.1365	0.5292	5.7241	0.5152	5.6448	0.7969	5.5169	0.5106
TailVaR(95)	5.7411	0.7721	6.3575	0.7419	6.4336	1.1772	6.1368	0.7419
TailVaR(99)	8.0594	2.1856	8.5371	2.0507	9.1671	3.3849	7.8591	2.0633

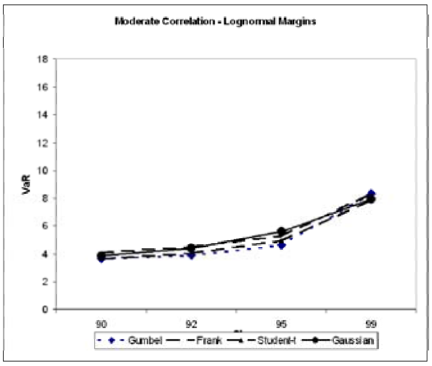
Table 5
Zero Dependence with Lognormal marginal distributions.

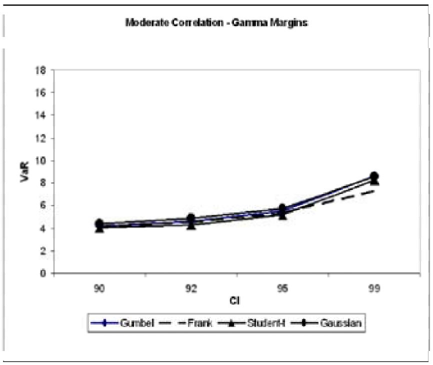
Copula:	Gumbel	Gumbel SE	Frank	Frank SE	Student-t	Student-t SE	Gaussian	Gaussian SE
VaR(90)	4.0402	0.1135	3.9328	0.1351	3.9634	0.1253	3.6528	0.1205
VaR(92)	4.3229	0.1287	4.1586	0.1485	4.1813	0.1429	3.9146	0.13185
VaR(95)	4.8326	0.1652	4.6963	0.1793	4.6816	0.1905	4.5056	0.16615
VaR(99)	6.2819	0.3560	6.6286	0.3866	6.3957	0.5342	6.1707	0.36805
TailVaR(90)	5.1197	0.2023	5.0583	0.2205	5.1388	0.3011	4.7064	0.20205
TailVaR(92)	5.3477	0.2339	5.3078	0.2511	5.4065	0.3555	4.9428	0.2334
TailVaR(95)	5.8217	0.3168	5.8586	0.3309	5.9735	0.5034	5.3991	0.3152
TailVaR(99)	7.4912	0.7669	7.4265	0.7663	8.3697	1.3278	6.7406	0.7706

Table 6
Zero Dependence with Gamma marginal distributions.

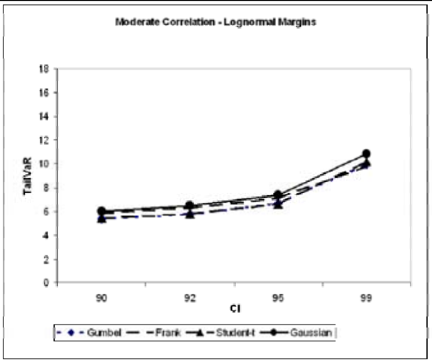
	Lognormal	Gamma

VaR





TailVaR



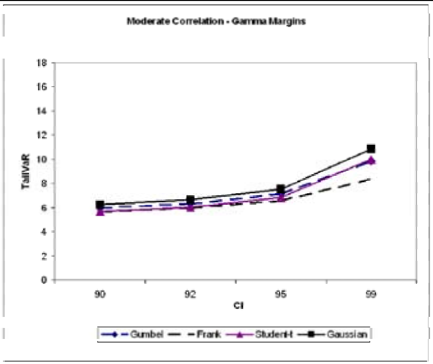


Table 7
Moderate Dependence with Lognormal and Gamma marginals.

Copula:	Gumbel	Gumbel SE	Frank	Frank SE	Student-t	Student-t SE	Gaussian	Gaussian SE
VaR(90)	3.6718	0.15735	4.1003	0.1485	3.6526	0.1523	3.8329	0.1529
VaR(92)	3.8864	0.1894	4.5158	0.1677	4.0943	0.17937	4.3875	0.1833
VaR(95)	4.6197	0.2672	5.2922	0.2182	4.965	0.2519	5.5983	0.2472
VaR(99)	8.0951	0.8819	7.9839	0.5978	7.4423	0.8369	7.9304	0.7213
TailVaR(90)	5.3818	0.7558	5.9029	0.4729	5.4423	0.7036	6.0082	0.5586
TailVaR(92)	5.7733	0.9156	6.3038	0.5656	5.8370	0.8491	6.4781	0.6721
TailVaR(95)	6.6966	1.3505	7.1477	0.8146	6.6432	1.2468	7.3759	0.9791
TailVaR(99)	9.9709	3.8686	9.8089	2.2552	10.1919	3.5619	10.8337	2.7687

Table 8
Moderate Dependence ($\rho = 0.5$) with Lognormal marginal distributions.

Copula:	Gumbel	Gumbel SE	Frank	Frank SE	Student-t	Student-t SE	Gaussian	Gaussian SE
VaR(90)	4.2202	0.1646	4.183	0.1492	4.0196	0.1560	4.3707	0.1488
VaR(92)	4.6448	0.1876	4.5688	0.1620	4.2671	0.1835	4.851	0.1699
VaR(95)	5.5191	0.2481	5.37	0.1973	5.1574	0.2429	5.7366	0.2216
VaR(99)	8.5211	0.5681	7.2875	0.3903	7.4673	0.5929	8.4296	0.5135
TailVaR(90)	5.9629	0.3344	5.6612	0.2261	5.6642	0.3394	6.2511	0.29022
TailVaR(92)	6.3520	0.3906	5.9789	0.2574	6.0362	0.3980	6.6559	0.3377
TailVaR(95)	7.1370	0.5383	6.5988	0.3398	6.8622	0.5498	7.4890	0.4622
TailVaR(99)	9.8133	1.3630	8.3458	0.8052	9.9656	1.3734	10.8409	1.1490

Table 9
Moderate Dependence ($\rho = 0.5$) with Gamma marginal distributions.

TailVaR increases with the probability level as expected.

(2) The estimates of the risk measures do not vary significantly by the copula assumed except for the TailVaR at high probability levels.

(3) The estimated standard errors for the risk measures increase with the probability level used in the risk measure. The assessment of the estimation error (standard errors) is important.

(4) The standard errors are in general higher than that with zero dependence.

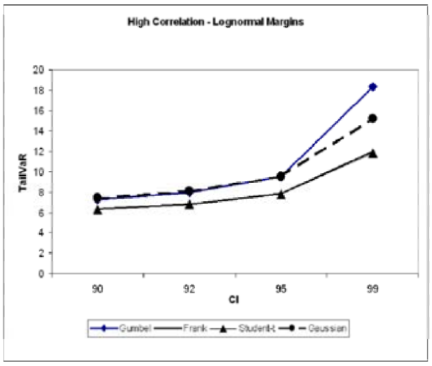
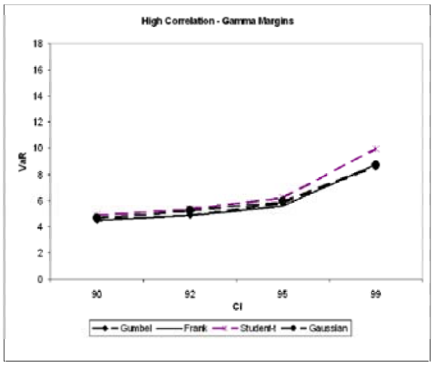
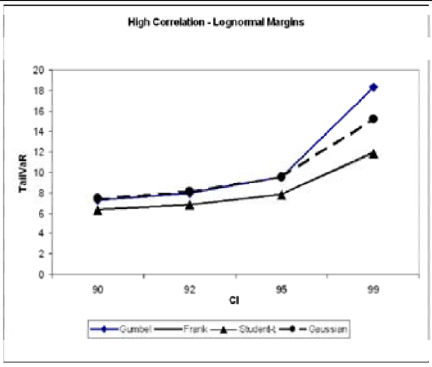
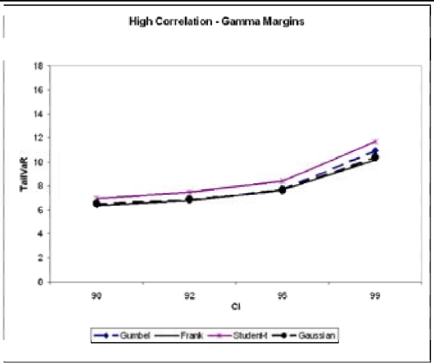
	Lognormal	Gamma
VaR		
TailVaR		

Table 10
High Dependence with Lognormal and Gamma marginals.

Copula:	Gumbel	Gumbel SE	Frank	Frank SE	Student-t	Student-t SE	Gaussian	Gaussian SE
VaR(90)	4.3808	0.1868	4.2727	0.1876	4.1744	0.1867	4.3991	0.1869
VaR(92)	4.7944	0.2231	4.6643	0.2213	4.6298	0.2128	5.1219	0.2172
VaR(95)	5.9687	0.3112	5.5546	0.3166	5.5485	0.3004	6.5042	0.3108
VaR(99)	9.7416	0.9518	8.597	0.7851	9.7665	0.9508	8.7271	1.01962
TailVaR(90)	7.2959	0.8706	6.3571	0.53848	6.3103	0.7439	7.4532	0.9491
TailVaR(92)	7.9625	1.0506	6.8285	0.6419	6.7970	0.8953	8.1452	1.1480
TailVaR(95)	9.5321	1.5366	7.8524	0.9234	7.7717	1.3063	9.5382	1.6896
TailVaR(99)	18.3013	4.3396	11.9929	2.5328	11.8423	3.6758	15.1932	4.8451

Table 11
High Dependence ($\rho = 0.9991$) with Lognormal marginal distributions.

Copula:	Gumbel	Gumbel SE	Frank	Frank SE	Student-t	Student-t SE	Gaussian	Gaussian SE
VaR(90)	4.5271	0.1893	4.474	0.2005	4.8932	0.1971	4.6748	0.1950
VaR(92)	4.8991	0.2141	4.8582	0.2247	5.4104	0.2239	5.2425	0.2150
VaR(95)	5.8523	0.2759	5.6179	0.2811	6.2417	0.2922	5.9359	0.2738
VaR(99)	8.5471	0.6304	8.597	0.4963	9.7665	0.6438	8.7271	0.6334
TailVaR(90)	6.4176	0.3737	6.3403	0.2775	6.9822	0.3668	6.5023	0.3445
TailVaR(92)	6.8357	0.4352	6.7568	0.3106	7.4415	0.4242	6.8896	0.3978
TailVaR(95)	7.7246	0.5967	7.6400	0.3918	8.4106	0.5718	7.6683	0.5411
TailVaR(99)	10.9541	1.4961	10.1686	0.8230	11.6815	1.3929	10.4171	1.3482

Table 12
High Dependence ($\rho = 0.9991$) with Gamma marginal distributions.

Table 13 gives the estimated confidence intervals for the VaR and TailVaR risk measure estimates. Estimated confidence intervals are given using both re-sampled bootstrap approach and the analytical approximation. The estimates of the standard error are reported for the 99% quantile (α) risk measures for a 95% (β) confidence interval. For VaR, the confidence intervals of the analytical approximations are determined using the approach given in section 7.1.1 and for the analytical approximations for TailVaR, the confidence intervals are determined using equations 2 to 4 given in section 7.1.2.

	Zero Dependence	Zero Dependence	Moderate Dependence	Moderate Dependence	High Dependence	High Dependence
Lognormal	CI for $VaR_{0.99}$	CI for $TailVaR_{0.99}$	CI for $VaR_{0.99}$	CI for $TailVaR_{0.99}$	CI for $VaR_{0.99}$	CI for $TailVaR_{0.99}$
Gaussian						
- Bootstrap	(5.80, 8.10)	(3.81, 11.90)	(6.51, 9.34)	(5.40, 16.26)	(8.97, 12.97)	(5.69, 24.68)
- Approximation	(6.44, 7.82)	(7.26, 8.44)	(7.43, 10.46)	(8.63, 13.02)	(9.15, 12.64)	(12.06, 18.32)
Student-t						
- Bootstrap	(5.84, 8.84)	(2.53, 15.80)	(5.80, 9.08)	(3.21, 17.17)	(6.75, 10.47)	(4.63,19.04)
- Approximation	(6.55, 7.942)	(7.85, 10.47)	(6.46, 10.20)	(8.35, 12.02)	(7.71, 11.05)	(9.51, 14.16)
Gumbel						
- Bootstrap	(5.56, 7.76)	(3.77, 12.34)	(6.36, 9.82)	(2.38, 17.55)	(7.87, 11.60)	(9.79, 26.80)
- Approximation	(5.66, 8.01)	(7.16, 8.95)	(7.02, 9.69)	(8.74, 11.19)	(8.40, 15.83)	(11.85, 24.74)
Frank						
- Bootstrap	(5.57, 7.88)	(4.51, 12.55)	(6.81, 9.15)	(5.38, 14.22)	(7.24, 10.32)	(7.02,16.95)
- Approximation	(6.30, 7.78)	(7.18, 9.88)	(7.43, 9.38)	(8.61, 11.00)	(7.96, 11.31)	(9.95, 14.03)
Gamma	CI for $VaR_{0.99}$	CI for $TailVaR_{0.99}$	CI for $VaR_{0.99}$	CI for $TailVaR_{0.99}$	CI for $VaR_{0.99}$	CI for $TailVaR_{0.99}$
Gaussian						
- Bootstrap	(5.44, 6.89)	(5.23, 8.25)	(7.42, 9.43)	(8.58, 13.09)	(7.48, 9.96)	(7.77, 13.05)
- Approximation	(5.58, 6.46)	(4.11, 9.36)	(7.25, 10.21)	(9.14, 12.53)	(7.84, 10.00)	(9.15, 11.68)
Student-t						
- Bootstrap	(5.34, 7.44)	(5.76, 10.97)	(6.30, 8.62)	(7.27, 12.65)	(8.50, 11.02)	(8.95, 14.41)
- Approximation	(5.93, 8.30)	(5.52, 11.21)	(6.50, 9.20)	(6.83, 13.10)	(8.99, 10.47)	(10.30, 13.05)
Gumbel						
- Bootstrap	(5.58, 6.97)	(5.98, 8.99)	(7.40, 9.63)	(7.14, 12.48)	(7.57, 9.51)	(8.02, 13.88)
- Approximation	(5.98, 7.06)	(4.78, 10.19)	(7.47, 9.09)	(8.81, 10.81)	(7.94, 10.17)	(9.07, 12.83)
Frank						
- Bootstrap	(5.87, 7.38)	(5.92, 8.92)	(6.52, 8.05)	(6.76, 9.92)	(7.36, 9.83)	(8.55, 11.78)
- Approximation	(6.03, 7.38)	(5.39, 9.45)	(6.97, 8.39)	(7.57, 9.11)	(7.97, 9.62)	(9.01, 11.31)

Table 13
Confidence Intervals from Bootstrap and Approximate Methods.

Although the assumed sample size is large for insurance data it is reasonable for asset returns data. However it is of interest to assess the improvement in the accuracy of the estimated risk measure from an increase in the sample size. To do this we re-simulate the four copulas assuming a sample size of 5000 bivariate samples for the case where the marginals are assumed to be Gamma and the dependence is moderate with $\rho = 0.5$ assuming $\alpha = 0.99$ for the risk measure and using $\beta = 0.95$ for the confidence interval. The results indiacte that:

- The confidence intervals for VaR under the analytical approximations are in similar for VaR compared with the more numerically intensive bootstrap approach.
- For TailVaR, the confidence intervals using the analytical approximation are similar and often less than the bootstrap approach.
- Increasing the sample size reduces the confidence intervals for both cases as expected, but they remain relatively large.
- The analytical approximation is quite accurate and more reliable for larger sample sizes.

Copula	$VaR_{0.99}$	$TailVaR_{0.99}$
Gaussian		
- Bootstrap (5000)	(7.72, 8.58)	(10.27, 14.26)
- Approximation	(7.41, 9.63)	(12.27, 12.50)
Student-t		
- Bootstrap (5000)	(7.79, 8.82)	(12.69, 15.12)
- Approximation	(7.47, 10.06)	(12.18, 13.21)
Gumbel		
- Bootstrap (5000)	(6.44, 9.86)	(8.94, 16.70)
- Approximation	(7.30, 9.63)	(12.47, 13.21)
Frank		
- Bootstrap (5000)	(6.76, 7.44)	(8.47, 11.31)
- Approximation	(6.68, 8.12)	(9.35, 10.43)

Table 14
Comparison of bootstrap with 1000 re-samples and the approximate method using 5000 sample size

7.6 *Diversification benefits*

An important practical issue for portfolios of risks is the extent of the diversification benefits from holding less than perfectly correlated risks. Risk portfolios with multiple lines of business can result in reductions in (regulatory) capital compared to stand alone risks because of the diversification benefits. Recent literature such as Tang and Valdez (2005) [15] and Dacorogna (2005) [4] study the impact of the dependence risks measures on the diversification effects for multi-line insurers. Dacorogna (2005)[4] demonstrates that the diversification benefits changes inversely with correlation in the context of DFA.

We assess the impact on the diversification benefits of the level of dependence for different copulas with different marginals. To begin with we define formally what is meant by a diversification benefit.

Following Dhaene et al (2002a [5], 2002b [6]), suppose that the random variables X_1 and X_2 are comonotonic (perfectly dependent), and S is the aggregate sum of the random variables X_1 and X_2 then:

$$VaR(Z) = VaR(X_1) + VaR(X_2)$$

and

$$TailVaR(Z) = TailVaR(X_1) + TailVaR(X_2)$$

If the businesses are dependent, then we can obtain diversification benefits by constructing a portfolio with total capital requirements determined on an aggregate basis. The diversification benefits for the VaR(S) and TailVaR(S) measures can be defined as below.

$$DB_{VaR} = 1 - \frac{VaR(S)}{VaR(Z)}$$

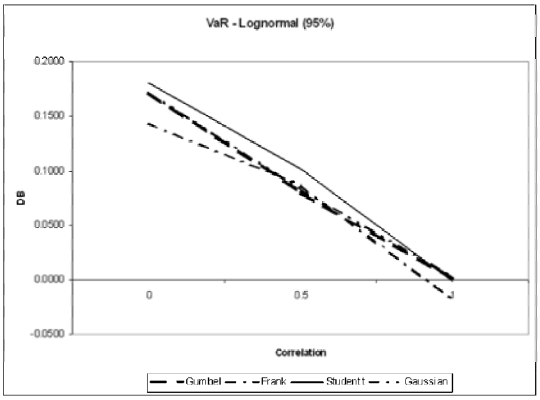
$$DB_{TailVaR} = 1 - \frac{TailVaR(S)}{TailVaR(Z)}$$

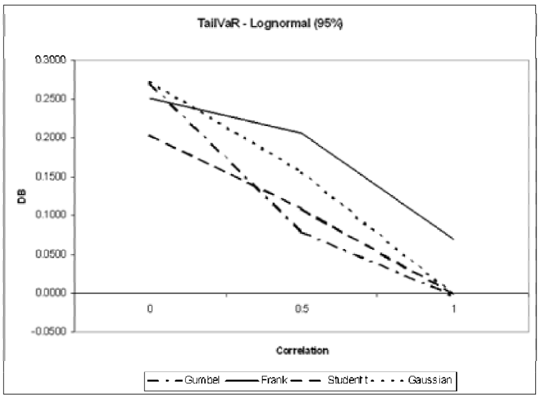
We compare diversification benefits relative to the comonotonic case. These differences should always be non-negative. However, there are circumstances where the values of DB_{VaR} are negative. This is due to the fact that VaR is not sub-additive.

To quantify the sensitivity of the diversification benefits for the different copulas to the changes in the dependence levels, we compare the values of VaR and TailVaR of the aggregate sum to the sum of the values of VaR and TailVaR on a stand alone basis. Tables 15, 16, 17 and 18 report the results of the comparison of the diversification benefits at two probability levels for both lognormal and gamma marginal distributions. The values for the risk measures were derived from 1000 simulations for each case. The case with correlation equal to 1 was estimated separately from the stand alone case and differs because of the different simulated values used in each case and the standard errors of the estimated values shown in the tables. Our results demonstrate that the diversification benefit decreases as the levels of dependence increases, as expected. This is consistent with results from Dacorogna (2005) [4]. Tables 15 to 18 demonstrate the sensitivity of the diversification benefits to the change of dependence for the different copulas and risk measures. The sensitivity measures (SM) of the diversification benefits to the correlation is determined as

$$SM = Abs[(\triangle DB/DB_{old})/(\triangle\rho/\rho_{old})]$$

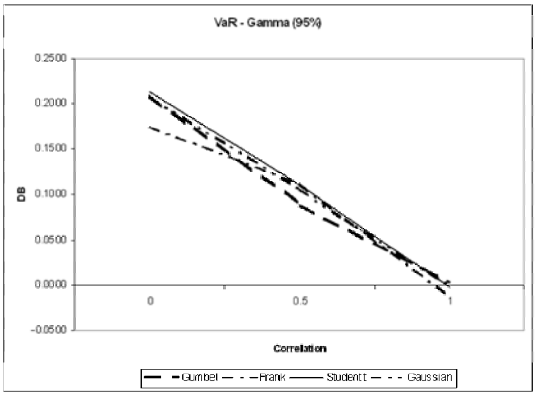
This measure is an elasticity measure. If SM is large, then the diversification benefit is relatively sensitive to the changes in dependence. This means that a small changes in dependence will have a proportionately large impact on the diversification benefits.

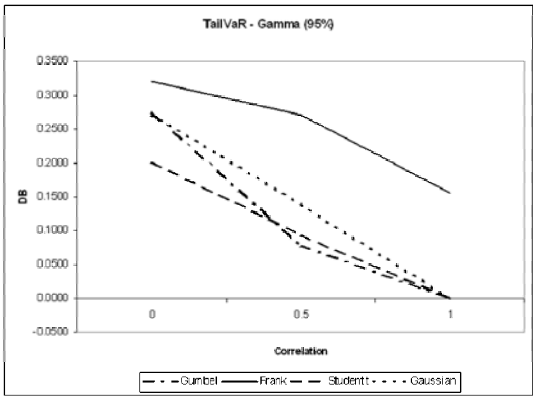




Dependence:	0	0.5	1.0	Standalone	Percentage Change (0 to 0.5)	Percentage Change (0.5 to 1)	Sensitivity Measure (0.5 to 1)
Gumbel							
VaR	4.61168	5.12214	5.56417	5.56894	-18%	-16%	0.99
TailVaR	7.06875	8.93574	9.73061	9.69805	-39%	-16%	1.04
Frank							
VaR	4.76896	5.08205	5.67033	5.56894	-11%	-21%	1.21
TailVaR	7.27179	7.6977	9.02292	9.69805	-9%	-27%	0.66
Student t							
VaR	4.56248	5.00516	5.57502	5.56894	-16%	-20%	1.01
TailVaR	7.72864	8.63863	9.70578	9.69805	-19%	-22%	1.01
Gaussian							
VaR	4.60657	5.10412	5.54911	5.56894	-18%	-16%	0.96
TailVaR	7.04907	8.1797	9.71435	9.69805	-23%	-32%	1.01

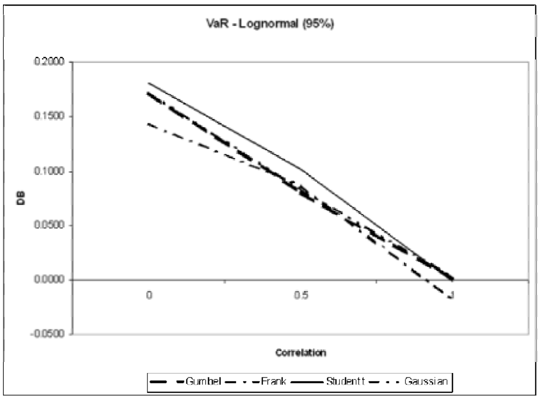
Table 15
VaR and TailVaR with Lognormal marginals at 95% probability.

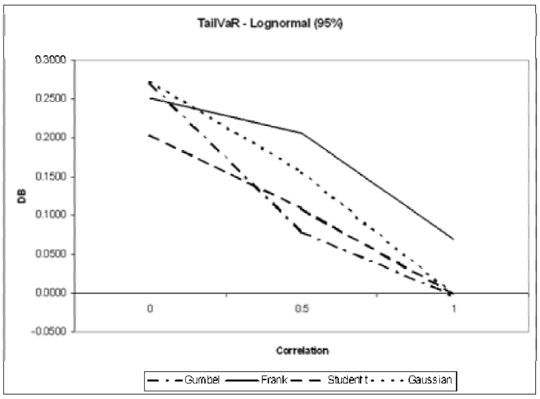




Dependence:	0	0.5	1.0	Standalone	Percentage Change (0 to 0.5)	Percentage Change (0.5 to 1)	Sensitivity Measure (0.5 to 1)
Gumbel							
VaR	4.74086	5.45965	5.97033	5.9908	-24%	-17%	0.96
TailVaR	6.22014	7.91653	8.58497	8.58367	-40%	-16%	1.0
Frank							
VaR	4.94274	5.32548	6.06552	5.9908	-13%	-25%	1.11
TailVaR	5.82998	6.25967	7.24418	8.58367	-10%	-23%	0.42
Student t							
VaR	4.71261	5.33089	6.00167	5.9908	-21%	-22%	1.02
TailVaR	6.8583	7.78444	8.58351	8.58367	-22%	-19%	1.0
Gaussian							
VaR	4.74659	5.35774	5.99424	5.9908	-20%	-21%	1.01
TailVaR	6.24575	7.3907	8.58153	8.58367	-27%	-28%	1.0

Table 16
VaR and TailVaR with Gamma marginals at 95% probability.

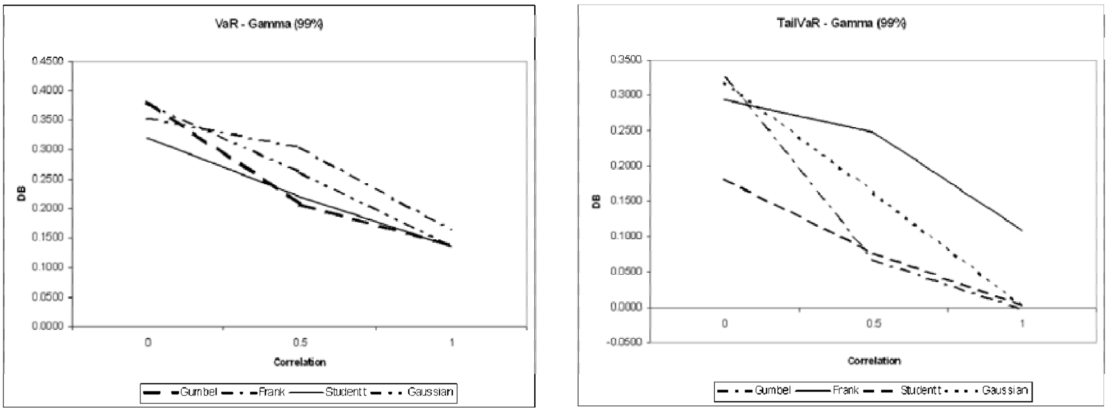




Dependence:	0	0.5	1.0	Standalone	Percentage Change (0 to 0.5)	Percentage Change (0.5 to 1)	Sensitivity Measure (0.5 to 1)
Gumbel							
VaR	7.22168	8.96562	9.82551	9.8392	-35%	-17%	0.98
TailVaR	10.73242	14.78677	16.0663	15.9596	-51%	-16%	1.09
Frank							
VaR	7.47496	7.94471	9.5919	9.8392	-10%	-33%	0.87
TailVaR	10.97305	11.54679	13.75618	15.9596	-7%	-28%	0.50
Student t							
VaR	7.75267	8.6941	9.83334	9.8392	-19%	-23%	0.99
TailVaR	12.61603	14.18707	15.9806	15.9596	-20%	-22%	1.01
Gaussian							
VaR	7.23463	8.39203	9.84707	9.8392	-24%	-30%	1.01
TailVaR	10.68378	12.76332	16.03861	15.9596	-26%	-41%	1.02

Table 17
VaR and TailVaR with Lognormal marginals at 99% probability.

Interestingly we find that the copula and marginal distribution assumptions do not have a major impact on the diversification benefits. The only case where this is not so is for the TailVaR risk measure at the high probability levels. The level of dependence is the main driving factor of the diversification benefits.



Dependence:	0	0.5	1.0	Standalone	Percentage Change (0 to 0.5)	Percentage Change (0.5 to 1)	Sensitivity
Gumbel							
VaR	6.60477	8.45984	9.19831	10.6786	-35%	-14%	0.33
TailVaR	8.20134	11.30551	12.16188	12.1224	-51%	-14%	1.05
Frank							
VaR	6.90903	7.43024	8.92775	10.6786	-10%	-28%	0.46
TailVaR	8.55037	9.12212	10.80508	12.1224	-9%	-28%	0.56
Student t							
VaR	7.26056	8.33346	9.23692	10.6786	-20%	-17%	0.39
TailVaR	9.92333	11.19752	12.10149	12.1224	-21%	-15%	0.98
Gaussian							
VaR	6.64307	7.88716	9.20978	10.6786	-23%	-25%	0.47
TailVaR	8.26123	10.15692	12.11177	12.1224	-31%	-32%	0.99

Table 18
VaR and TailVaR with Gamma marginals at 99% probability.

8 Summary of Results

The main results of this study are:

- The estimates of VaR and TailVaR increase with the level of dependence as would be expected but the effect of the copula is not as important as the marginal distributions and the level of dependence except in the case of the TailVaR risk measure for high probability levels.
- Risk measures in practice are estimates. The standard errors of these estimates, even for large sample sizes, are quite large. Regulators should take into account estimation errors in setting risk base capital requirements.measure used and the level of dependence.
- The standard errors of the estimates increase with the probability level used in the risk measure and are higher for the TailVaR risk measure.
- As expected, the estimated confidence intervals for the VaR and TailVaR risk measure reduce

with an increase in sample size but remain significant in size even for reasonable large sample sizes.

- The analytical approximation for the standard errors of the risk measures are reasonable for assessing the accuracy of risk measure estimates reducing the need for the high computational requirements of bootstrap estimates.
- The level of dependence has a significant impact on the diversification benefits in contrast to the copula used. Estimation of the marginal distribution and overall levels of dependence are important in determining risk baaed economic capital using VaR or TailVaR.

9 Conclusion

Dependence for risk measures is an important factor in risk based capital assessment for banks and insurance companies with multiple lines of business. The advantage of a diversifying risk portfolio is the reduced regulatory capital arising from diversification benefits. There is an incentive for companies to properly measure and account for the dependence amongst risks. Copulas have become an accepted method for assessing capital requirements for dependent risks. Insurance risks are in general positively correlated and the level of dependence is important in assessing risk based capital. We have used experiments for bivariate dependence models using different copulas, different marginal distributions and different levels of dependence to quantify the impact of these factors on risk based capital using VaR and TailVaR risk measures. We have assessed the accuracy of the estimated risk measures by quantifying standard errors of the estimates.
The results of the case study experiments interestingly demonstrate that the choice of copula does not have such a significant impact on the estimated risk measures except for the case of the TailVaR risk measure at high probability levels. The overall level of dependence and the marginal distribution are more important in determining the estimated risk measure. Even for large sample sizes, the standard errors of the estimated risk measure are large and this should be taken into account in establishing risk based capital. The analytical approximation we have used for the standard errors of the risk measure estimates performs reasonably well and can be used to save the computational effort required for a more extensive bootstrap approach.

References

[1] Artzner, P., 1999, Application of Coherent Risk Measures to Capital Requirements in Insurance, *North American Actuarial Journal* 3 (2), 258-277.

[2] Bradley, B.O. and Taqqu, M.S., 2003, Financial Risk and Heavy Tails, in S.T. Rachev (ed.) *Handbook of Heavy-tailed Distributions in Finance*, North Holland.

[3] Cherubini, U., Luciano, E., and Vecchiato, W., 2004, *Copula Methods in Finance*, Wiley Finance.

[4] Dacorogna, M., 2005, Risk Aggregation and Dependence Structure, in *Capital Management in the Reinsurance Industry.*

[5] Dhaene, J., Denuit, M., Goovaerts, M. J., Kaas, R., and Vyncke, D., 2001b, The concept of comonotonicity in actuarial science and finance: Theory. *Insurance: Mathematics and Economics*, 31(1): 3-33.

[6] Dhaene, J., Denuit, M., Goovaerts, M. J., Kaas, R., and Vyncke, D., 2001a, The concept of comonotonicity in actuarial science and finance: Application. *Insurance: Mathematics and Economics*, 31(1): 133-161.

[7] Embrechts, P., Lindskog, F. and McNeil, A., 2001, Modelling dependence with copulas and applications to risk management, *Handbook of heavy tailed distributions in finance*, edited by Rachev St., Elsevier, North Holland, Amsterdam

[8] Embrechts, P., Resnick, S. and Samorodnitsky, G., 1999, Extreme Value theory as a risk management tool,*North American Actuarial Journal*, Vol. 3(2)

[9] Frees, E. and Valdez, E., 1999, Understanding Relationships Using Copulas, *North American Actuarial Journal*, Vol. 2,1-25.

[10] Hardy, M., 2005, Simulation VaR and CTE, $www.fenews.com/fen47/topics_act_analysis/topics-act-analysis.htm.$

[11] Isaacs, D., 2003, Capital Adequacy and Dependence, *Institute of Actuaries of Australia XIV General Insurance Seminar*, 229-249.

[12] Lindskog, F., 2000, *Modelling Dependence with Copulas and Application to Risk Management*, ETH Working Paper, Zurich.

[13] Nelsen, R.B., 1999, *An Introduction to Copulas*, Lecture Notes in Statist. 139., New York: Springer- Verlag.

[14] Tang, A., 2005, *Economic Capital and the Aggregation of Risks using Copulas*, University of New South Wales, Actuarial Studies Honours Thesis.

[15] Tang, A. and Valdez, E. A., 2005, *Economic Capital and the Aggregation of Risks Using Copulas*, University of New South Wales, Actuarial Studies Working Paper.

[16] Venter, G., 2002, Tails of Copulas, *CAS Proceedings*, Vol. LXXXIX, 68-113.